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Entropy, Reversible Diffusion Processes, and Markov Uniqueness

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Consider a symmetric bilinear form $\mathscr{E}_\varphi$ defined on $\mathscr{C}_c^\infty(\mathbb{R}^d)$ by

$$\mathscr{E}_\varphi(f, g) = \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \, \varphi^2 \, dx, \qquad \varphi \in H^1_{loc}(\mathbb{R}^d)$$

In this paper we study the stochastic process associated with the smallest closed markovian extension of $(\mathscr{E}_\varphi, \mathscr{C}_c^\infty)$, and give a new proof of Markov uniqueness (i.e. the uniqueness of a closed markovian extension) based on purely probabilistic arguments. We also give another purely analytic one. As a consequence, we show that all invariant measures are reversible, provided they are of finite energy. The problem of uniqueness of such measures is also partially solved.

1. INTRODUCTION

Consider the symmetric bilinear form $\mathscr{E}_\varphi$ defined on $\mathscr{C}_c^\infty(\mathbb{R}^d)$ by

$$\mathscr{E}_\varphi(f, g) = \int \nabla f \cdot \nabla g \, \varphi^2 \, dx$$

for $\varphi \in H^1_{loc}(\mathbb{R}^d)$. This form is closable and its minimal extension $(\mathscr{E}_\varphi, H^1_o(\varphi^2 \, dx))$ is actually a Dirichlet form. Several questions are then natural to ask:

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1. does it exist a maximal closed markovian extension $(\mathscr{E}^+_\varphi, \mathscr{D}(\mathscr{E}^+_\varphi))$ of $(\mathscr{E}_\varphi, \mathscr{C}^\infty_c)$, and can we describe explicitly this extension?

2. when is $(\mathscr{E}^+_\varphi, \mathscr{D}(\mathscr{E}^+_\varphi))$ equal to $(\mathscr{E}_\varphi, H^1_o(\varphi^2\,dx))$? (this property is known as Markov uniqueness)

3. is it possible to describe the (eventually) associated Markov process, in particular in terms of a Girsanov transform of a standard Brownian motion?

When $\varphi \equiv 1$, it is well known that Markov uniqueness holds with $\mathscr{D}(\mathscr{E}^+_\varphi) = H^1(\mathbb{R}^d)$, the usual Sobolev space, since $H^1(\mathbb{R}^d) = H^1_o(\mathbb{R}^d)$. When $\varphi \not\equiv 1$, the three questions above have been studied for a long time by many authors. Surprisingly, they only recently received a fully satisfactory answer (see [AHKS77] [AR89] [AKR90] [RZ92] [RZ94] [Son94b] [Son94a], ...) namely

$$\mathscr{D}(\mathscr{E}^+_\varphi) \text{ is a “pseudo” Sobolev space and Markov uniqueness holds.} \tag{1.1}$$

The stochastic structure of the associated process follows from general arguments in the theory of Dirichlet forms, and is studied in [ARZ93a].

In this paper, we shall link these questions to the construction (and the properties) of singular diffusions (more precisely Brownian motions with singular drifts) which have deserved attention for some times, in particular in Nelson's stochastic approach of the Schrödinger equation. The recent new developments of this theory allow to extend the results presented here to more general situations, like general $\mathbb{R}^d$-valued diffusion processes (with a non necessarily uniformly elliptic diffusion matrix), or infinite dimensional diffusion processes. One can also expect to get interesting results in the non symmetric case.

Let us describe briefly the contents of the present paper, and compare our results (or proofs) with those of the (impressive) existing literature.

In Section 2, we introduce the main tools and results concerning the Dirichlet form associated with $\mathscr{E}_\varphi$. In particular, we take up the friendly challenge proposed to us by M. Röckner, and give a completely elementary (and purely analytical) proof of Markov uniqueness (assuming the maximality results due to [AKR90]) (see theorem 2.7). Actually, even in this simple case, the proofs proposed in the existing literature are a sophisticated mixture of deep functional analytical results and a touch of Probability theory ([RZ92], [RZ94], [Son94b]), or use the specific Gaussian structure of the underlying Brownian motion ([Son94b] and [Son94a]).

Section 3 collects some results on stationnary singular diffusion processes associated with the generator $S_\varphi = \frac{1}{2}\Delta + (\nabla\varphi/\varphi)\cdot\nabla$. These processes (in the non-stationnary case) appeared in the pioneering work by Nelson [Nel88] on stochastic mechanics. A first existence result is due to Carlen [Car84]. A new completely different approach for existence was proposed by Föllmer ([Föl88]), at least in implicit form). This approach is based on relative entropy, and was recently developed by Léonard and one of the authors (see [Cl94], [CL95a] and [CL95b]). It allows in particular to build a Brownian motion with drift $\nabla\varphi/\varphi$, and to characterize it as a solution of a minimization problem. We shall apply these results here in the much more simpler symmetric case (see Theorem 3.4 and Proposition 6.1). Time reversibility is obtained thanks to results due to Föllmer [Föl84] (see also [CP95] for extensions). The analytic building (in the symmetric case) of this process is explained in [AR91].

In Section 4, we briefly study the fine (in the usual potential theoretic sense) structure of the diffusion process builded in Section 3. The main result is that the nodal set $\{\varphi = 0\}$ cannot be attained, starting from any point outside of it (up to a nice modification of φ). This result is a consequence of Nelson's estimate, as recalled in [MZ85], and an ad-hoc choice of φ. It can easily be extended to more general frames.

In Section 5, we link the singular diffusion process to the Dirichlet form. The uniqueness of a quasi-regular extension of $(\mathscr{E}_\varphi, \mathscr{C}_c^\infty)$ is then an immediate consequence of old results on martingale problems explained in [Jac79]. The unattainability of the nodal set gives the key argument in Song's proof of regularity (i.e. of Markov uniqueness, assuming the maximality result [Son94b]). This proof is simplified in order to get a purely probabilistic one (the only analytical material required is really elementary and does not call to Dirichlet forms theory).

Finally, in Section 6, we present some immediate consequences to the study of invariant measures of finite energy. These results extend (in the framework of the present paper) similar results of Bogachev and Röckner ([BR94a] and [BR94b]), and are actually the probabilistic counterpart of the methods of these authors.

For the sake of shortness, some straightforward proofs are not given. The interested reader shall find them, as well as extensions to general diffusion processes in $\mathbb{R}^d$, in [Fra95a].

2. Dirichlet Forms

We first study by purely analytical means the Dirichlet form we introduce above. Though we shall sometimes recall basic definitions, we refer to Fukushima's book [Fuk80] for all general results on Dirichlet forms in $\mathbb{R}^d$.

Consider the symmetric form $\mathscr{E}_\varphi$ defined for f and g in $\mathscr{C}_c^\infty$ by

$$\mathscr{E}(f, g) = \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \, \varphi^2 \, dx \tag{2.1}$$

For this definition to make sense, we have to assume that $\varphi \in L^2_{loc}(dx)$, but if we want to mimic what is done in the classical case ($\varphi \equiv 1$), and for reasons which will be clear later, we shall assume that $\varphi \in H^1_{loc}(dx)$. Actually, this assumption is necessary and sufficient for $\mathscr{E}_\varphi$ to be admissible in the sense of [AR89], because of the following result:

PROPOSITION 2.1 (see [ARZ93b, Prop. 1.5]). *Let ν be a finite positive measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ s.t. for every $i=1, ..., d$ there exists $\beta_i \in L^2(d\nu)$ s.t.*

$$\forall f \in \mathscr{C}_c^\infty \int \nabla_i f \, d\nu = \int f\beta_i \, d\nu$$

Then $\nu = \varphi^2 \, dx$ for some $\varphi \in H^1(dx)$.

The following properties of $\mathscr{E}_\varphi$ have been proved to be true:

PROPOSITION 2.2. (i) *The form $(\mathscr{E}_\varphi, \mathscr{C}_c^\infty)$ is closable in $L^2(\varphi^2 \, dx)$ see* [MR92]

(ii) *Existence and description of its maximal closed markovian extension see* [AKR90] *and* [TAK92]

(iii) *Markov uniqueness i.e. uniqueness of a markovian closed extension see* [RZ92] *and* [RZ94].

Our aim is to give elementary proofs of parts (i) and (iii) of this Proposition 2.2.

The first difficulty is the following: An element f of $L^2(\varphi^2 \, dx)$ does not necessarily belong to $\mathscr{D}'$ (because φ^2 is not bounded by below) and so the usual ∇f is not a priori well defined. Nevertheless for such an f, $f\varphi^2$ belongs to $L^1_{loc}(dx)$ and so $\nabla(f\varphi^2)$ is well defined in $\mathscr{D}'$. Furthermore, if $f \in \mathscr{C}_c^\infty$

$$\nabla(f\varphi^2) = \varphi^2 \, \nabla f + 2f\varphi \, \nabla\varphi \qquad (\in (L^1(dx))^d) \tag{2.2}$$

So it is natural to define the weighted Sobolev space $H^1(\varphi^2 \, dx)$ as follows (see [ARZ93b] Lemma 2.2):

DEFINITION 2.3. *$H^1(\varphi^2 \, dx)$ is the set of elements $f \in L^2(\varphi^2 \, dx)$ s.t. there exists an element $\bar{\nabla} = (\bar{\nabla}_i f)_{i \in \{1 ..., d\}} \in (L^2(\varphi^2 \, dx))^d$ satisfying*

$$\forall i \in \{1, ..., d\} \qquad \varphi^2 \, \bar{\nabla}_i f = \nabla_i (f\varphi^2) - 2f\varphi \, \nabla_i \varphi \qquad \text{in } \mathscr{D}'$$

that is, the following integration by parts formula

$$\int g \, \bar{\nabla}_i f\varphi^2 \, dx = -\int f \nabla_i g \, \varphi^2 \, dx - 2 \int f g \, \frac{\nabla_i \varphi}{\varphi} \, \varphi^2 \, dx \tag{2.3}$$

holds for all $i \in \{1, ..., d\}$ and $g \in \mathscr{C}_c^\infty$.

Notice that if $f \in H^1(\varphi^2\,dx)$ then $f\varphi^2 \in W^{1,1}_{loc}$ (usual Sobolev space). The elementary proof of the next result is left to the reader:

PROPOSITION 2.4. *The form* $(\mathscr{E}_\varphi, H^1(\varphi^2\,dx))$ *defined by*

$$\forall f, g \in H^1(\varphi^2\,dx) \qquad \mathscr{E}_\varphi(f, g) = \int \bar{\nabla} f \cdot \bar{\nabla} g\, \varphi^2\,dx$$

is a closed extension of $(\mathscr{E}_\varphi, \mathscr{C}_c^\infty)$, *which in turn is closable.*

Since $(\mathscr{E}_\varphi, \mathscr{C}_c^\infty)$ admits a closed extension, it is closable and we shall of course denote its smallest closed extension by $(\mathscr{E}_\varphi, H^1_o(\varphi^2\,dx))$, which is actually a Dirichlet form since $(\mathscr{E}_\varphi, \mathscr{C}_c^\infty)$ is trivially markovian. The Markov property of $(\mathscr{E}_\varphi, H^1(\varphi^2\,dx))$ is not so immediate in view of its definition. As in the classical case, one can get other descriptions of $(\mathscr{E}_\varphi, H^1(\varphi^2\,dx))$ in order to prove the Markov property. This description was obtained in [AR90] Th 3.2, [AKR90] Prop 2.2, [RZ92] and [ARZ93b] Lemma 2.2 in this and more general contexts (see also [Son94b]).

PROPOSITION 2.5. $f \in H^1(\varphi^2\,dx)$ *if and only if* $f \in L^2(\varphi^2\,dx)$ *and for all* $k \in \mathbb{R}^d$, ν_k-*almost all* $x \in K^\perp$ (*where* ν_k *is the image of* $\varphi^2\,dx$ *by the projection* $\mathbb{R}^d \to k^\perp$) *the following holds*:

(i) $\left\{\begin{matrix}\mathbb{R} \to \mathbb{R} \\ s \to f(x+sk)\end{matrix}\right\}$ *has an absolutely continuous ds-version* $\tilde{f}_x$ *on the set* $\{s/\varphi^2(x+sk) > 0\}$.

(ii) $d\tilde{f}_x/ds \in L^2(\varphi^2\,dx)$.

In this case if we define $\bar{\partial} f/\partial k := d\tilde{f}_x/ds$, *then* $\bar{\nabla}_i f = \bar{\partial} f/\partial e_i$.

Here are some important consequences, already noticed by Albeverio and Röckner:

COROLLARY 2.6. (i) $(\mathscr{E}_\varphi, H^1(\varphi^2\,dx))$ *is markovian* (*hence a Dirichlet form*). *It is an easy consequence of the chain rule and proposition* 2.5.

(ii) *If* f *and* g *belongs to* $H^1(\varphi^2\,dx)$ *and if* fg, $f\bar{\nabla}_i g$ *and* $g\bar{\nabla}_i f$ *are in* $L^2(\varphi^2\,dx)$ *for all* $i \in \{1, ..., d\}$, *then* $fg \in H^1(\varphi^2\,dx)$ *and* $\bar{\nabla}(fg) = f\bar{\nabla} g + g\bar{\nabla} f$.

(iii) *The set of bounded compactly supported functions of* $H^1(\varphi^2\,dx)$ *is a dense subset of* $H^1(\varphi^2\,dx)$ *for the norm* $\sqrt{\mathscr{E}^1_\varphi}$. *It is an immediate consequence of i and ii, by using a truncature argument.*

We now are ready to state the main result of [RZ94] and to give an elementary proof of it.

THEOREM 2.7. *$\mathscr{C}_c^\infty$ is $\mathscr{E}_\varphi^1$ dense in $H^1(\varphi^2\,dx)$, i.e., $H_o^1(\varphi^2\,dx) = H^1(\varphi^2\,dx)$.*

Proof. According to corollary 2.6, it is enough to approximate functions in $H^1(\varphi^2\,dx)$ which are bounded and compactly supported. In the sequel, f will denote such a function.

Let J_ε be a standart mollifier, i.e. for every $\varepsilon \in]0, 1]$, $J_\varepsilon = 1/\varepsilon^d\, J(x/\varepsilon)$ where $J \in \mathscr{C}_c^\infty$ satisfies $0 \leqslant J \leqslant 1$, supp $J \subset B(0, 1)$ and $\int J(x)\,dx = 1$.

Since f is bounded with compact support, $J_\varepsilon * f$ is well defined $\mathscr{C}^\infty$ compactly supported with $\operatorname{supp} J_\varepsilon * f \subset \operatorname{supp} f + \varepsilon B(0, 1)$ and bounded by $\|f\|_\infty$ for all $\varepsilon > 0$. We shall prove that

$$\text{for a given sequence } (\varepsilon_n)_n \text{ decreasing to 0, } J_{\varepsilon_n} * f \text{ converges to } f \text{ in } H^1(\varphi^2\,dx) \tag{2.4}$$

The easy part is the $L^2(\varphi^2\,dx)$-convergence.

LEMMA 2.8. *One can find a sequence $(\varepsilon_n)_n$ decreasing to 0, such that $J_{\varepsilon_n} * f$ converges to f in $L^2(\varphi^2\,dx)$.*

Proof of Lemma 2.8. First, $J_\varepsilon * f \in L^2(dx)$ and converges to f in $L^2(dx)$ when ε goes to 0. Thus we can find a sequence $(\varepsilon_n)_n$ such that $J_{\varepsilon_n} * f$ converges to $f\,dx$ a.e. Finally, since

$$|\varphi(J_{\varepsilon_n} * f) - \varphi f|^2 \leqslant 2\,\|f\|_\infty^2\,|\varphi|^2\,\mathbb{1}_{(\operatorname{supp} f + B(0,1))}$$

which belongs to $L^1(dx)$, one may apply Lebesgue's bounded convergence theorem. ∎

We now have to prove that $\nabla_i(J_{\varepsilon_n} * f)$ converges to $\overline{\nabla} f$ in $L^2(\varphi^2\,dx)$ for a given sequence $(\varepsilon_n)_n$, or equivalently that $\varphi\nabla_i(J_{\varepsilon_n} * f)$ converges to $\varphi\overline{\nabla}_i f$ in $L^2(dx)$. But

$$\begin{aligned} \|\varphi\nabla_i(J_\varepsilon * f) - \varphi\overline{\nabla}_i f\|_{L^2(dx)}^2 &\leqslant \|\varphi\overline{\nabla}_i f - (J_\varepsilon * (\varphi\overline{\nabla}_i f))\|_{L^2(dx)}^2 \\ &\quad + \|J_\varepsilon * (\varphi\overline{\nabla}_i f) - \varphi\nabla_i(J_\varepsilon * f)\|_{L^2(dx)}^2 \end{aligned} \tag{2.5}$$

since $\varphi\overline{\nabla}_i f \in L^2(dx)$. But the first term of the above sum goes to 0 with ε. So we only have to study the second term. To this end we adapt the idea of [Fra94] Th. 2.1.8. We write

$$\|\varphi\nabla_i(J_\varepsilon * f) - (J_\varepsilon * (\varphi\bar{\nabla}_i f))\|^2_{L^2(dx)}$$

$$= \int \left| \varphi(x) \int \nabla_i J_\varepsilon(x-y) f(y)\, dy - \int J_\varepsilon(x-y)\, \varphi(y)\, \bar{\nabla}_i f(y)\, dy \right|^2 dx$$

$$= \int \bigg| \int \nabla_i J_\varepsilon(x-y) f(y)\, [\varphi(x) - \varphi(y)]\, dy$$

$$+ \int [\nabla_i J_\varepsilon(x-y) f(y)\, \varphi(y) - J_\varepsilon(x-y)\, \varphi(y)\, \bar{\nabla}_i f(y)\, dy \bigg|^2 dx \qquad (2.6)$$

and we apply the following lemma:

Lemma 2.9. *For all* $g \in \mathscr{C}_c^\infty$

$$\int \bar{\nabla}_i f\, \varphi\, g\, dx + \int f \nabla_i \varphi\, g\, dx + \int f\, \varphi\, \nabla_i g\, dx = 0$$

The proof of Lemma 2.9 is postponed to the end of the section. Just remark that Lemma 2.9 is meaningful since $\varphi\bar{\nabla}_i f$ and φf are in $L^2(dx)$, so in $L^1_{loc}(dx)$ while $f\nabla_i\varphi \in L^1_{loc}(dx)$ since f is bounded and $\nabla_i\varphi \in L^2_{loc}(dx)$.

Applying Lemma 2.9 with $g(y) = J_\varepsilon(x-y)$ in (2.6), we get

$$\|\varphi\nabla_i(J_\varepsilon * f) - (J_\varepsilon * (\varphi\bar{\nabla}_i f))\|^2_{L^2(dx)}$$

$$= \int \bigg| \int \nabla_i J_\varepsilon(x-y) f(y) [\varphi(x) - \varphi(y)]\, dy$$

$$+ \int J_\varepsilon(x-y) f(y)\, \nabla_i(\varphi)(y)\, dy \bigg|^2 dx$$

$$\leqslant 2\, \|J_\varepsilon * (f\nabla_i\varphi)\|^2_{L^2(dx)} + 2 \int \left| \int \nabla_i J_\varepsilon(x-y) f(y) [\varphi(x) - \varphi(y)]\, dy \right|^2 dx$$

$$\leqslant 2\, \|f\nabla_i\varphi\|^2_{L^2(dx)} + 2 \int \left| \int \nabla_i J_\varepsilon(x-y) f(y) [\varphi(x) - \varphi(y)]\, dy \right|^2 dx \quad (2.7)$$

To control the second term, replace φ by $\hat{\varphi} \in H^1$ defined by:

$$\hat{\varphi} = \varphi\xi \quad \text{with} \quad \xi \in \mathscr{C}_c^\infty \quad \text{and} \quad \mathbb{1}_{\operatorname{supp} f + B(0,2)} \leqslant \xi \leqslant \mathbb{1}_{\operatorname{supp} f + B(0,3)}$$

Then

$$
\begin{aligned}
&\int \left| \int \nabla_i J_\varepsilon(x-y) f(y) [\varphi(x)-\varphi(y)]\, dy \right|^2 dx \\
&\quad = \int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J(s) f(x+s\varepsilon) \frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon}\, ds \right|^2 dx \\
&\quad \leqslant 2 \int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J(s) f(x+s\varepsilon)(-\nabla\hat{\varphi}(x+s\varepsilon)\cdot s)\, ds \right|^2 dx \\
&\qquad + 2 \int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J(s) f(x+s\varepsilon) \right. \\
&\qquad\quad \left. \times \left[\frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon} + \nabla\hat{\varphi}(x+s\varepsilon)\cdot s \right] ds \right|^2 dx
\end{aligned} \tag{2.8}
$$

Of course

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J(s) f(x+s\varepsilon)(-\nabla\hat{\varphi}(x+s\varepsilon)\cdot s)\, ds \right|^2 dx \\
&\quad \leqslant C^{ste} \, \|\nabla_i J\|_\infty^2 \sum_{j=1}^{d} \|f \nabla_j \varphi\|_{L^2(dx)}^2
\end{aligned} \tag{2.9}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J(s) f(x+s\varepsilon) \left[\frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon} + \nabla\hat{\varphi}(x+s\varepsilon)\cdot s \right] ds \right|^2 dx \\
&\quad \leqslant C^{ste} \, \|\nabla_i J\|_\infty^2 \, \|f\|_\infty^2 \\
&\qquad \times \int_{\mathbb{R}^d} \int_{B(0,1)} \left| \frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon} + \nabla\hat{\varphi}(x+s\varepsilon)\cdot s \right|^2 ds\, dx \\
&\quad \leqslant C^{ste} \, \|\nabla_i J\|_\infty^2 \, \|f\|_\infty^2 \int_{B(0,1)} \left\| \frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon} + \nabla\hat{\varphi}(x+s\varepsilon)\cdot s \right\|_{L^2(dx)}^2 ds
\end{aligned} \tag{2.10}
$$

The quantity under the integral goes to 0 as ε goes to 0. By using for instance Fourier transforms, it is easy to see that the convergence is uniform in s on the closed unit ball. We thus have proved

LEMMA 2.10.

$$\|\varphi\nabla_i(J_\varepsilon * f) - (J_\varepsilon * (\varphi\bar{\nabla}_i f))\|^2_{L^2(dx)} \leqslant C^{ste} \sum_{j=1}^{d} \|f\nabla_j\varphi\|^2_{L^2(dx)} + C^{ste}\,\|f\|^2_\infty\,\theta(\varepsilon)$$

with $\theta(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$, *and* θ *only depends on* $\hat{\varphi}$, *thus on* $\operatorname{supp} f$.

Of course we want to show that the left hand side in Lemma 2.10 goes to 0 with ε. To this end, we shall use one more approximation.

Take $f_\eta = J_\eta * f$ which is $\mathscr{C}_c^\infty$. By the same argument as in Lemma 2.8 we can find a sequence $(\eta_k)_k$ such that

$$\sum_{j=1}^{d} \|(f - f_{\eta_k})\,\nabla_j\varphi\|^2_{L^2(dx)} \xrightarrow[k\to+\infty]{} 0 \tag{2.11}$$

Hence

$$\begin{aligned}
&\|\varphi\nabla_i(J_\varepsilon * f) - (J_\varepsilon * (\varphi\bar{\nabla}_i f))\|^2_{L^2(dx)} \\
&\qquad\leqslant 2\,\|\varphi\nabla_i(J_\varepsilon * (f - f_{\eta_k})) - (J_\varepsilon * (\varphi\bar{\nabla}_i(f - f_{\eta_k})))\|^2_{L^2(dx)} \\
&\qquad\quad + 2\,\|\varphi\nabla_i(J_\varepsilon * f_{\eta_k}) - (J_\varepsilon * (\varphi\bar{\nabla}_i f_{\eta_k}))\|^2_{L^2(dx)} \\
&\qquad\leqslant C^{ste} \sum_{j=1}^{d} \|(f - f_{\eta_k})\,\nabla_j\varphi\|^2_{L^2(dx)} + C^{ste}\,\|(f - f_{\eta_k})\|^2_\infty\,\theta(\varepsilon) \\
&\qquad\quad + 2\,\|\varphi\nabla_i(J_\varepsilon * f_{\eta_k}) - (J_\varepsilon * (\varphi\bar{\nabla}_i f_{\eta_k}))\|^2_{L^2(dx)}
\end{aligned} \tag{2.12}$$

using Lemma 2.10 with f replaced by $(f - f_{\eta_k})$ (we can do it without changing anything in the proof of this lemma since we actually have $\hat{\varphi} = \varphi$ on $\operatorname{supp} f + B(0, 2)$, thus on $\operatorname{supp}(f - f_{\eta_k}) + B(0, 1)$, which is enough for (2.8) to hold). By choosing first η_k and then ε, the first two terms in the above sum can be chosen arbitrarily small (since $\|(f - f_{\eta_k})\|^2_\infty \leqslant 2\,\|f\|^2_\infty$). It remains to show that for a fixed η_k the third term goes to 0 with ε i.e. that

$$\forall g \in \mathscr{C}_c^\infty\ \|\varphi\nabla_i(J_\varepsilon * g) - (J_\varepsilon * (\varphi\nabla_i g))\|^2_{L^2(dx)} \xrightarrow[\varepsilon\to 0]{} 0 \tag{2.13}$$

But

$$\begin{aligned}
&\|\varphi\nabla_i(J_\varepsilon * g) - (J_\varepsilon * (\varphi\nabla_i g))\|^2_{L^2(dx)} \\
&\quad = \int \left| \int J_\varepsilon(x-y)\,\nabla_i g(y)[\varphi(x) - \varphi(y)]\,dy \right|^2 dx \\
&\quad \leqslant C^{ste}\,\|J\|^2_\infty\,\|\nabla_i g\|^2_\infty \int_{B(0,1)} \|(\varphi\xi_g)(\cdot) - (\varphi\xi_g)(\cdot + s\varepsilon)\|^2_{L^2(dx)}\,ds
\end{aligned} \tag{2.14}$$

(by the same manipulations as in (2.8), with $\xi_g \in \mathscr{C}_c^\infty$ and $\xi_g = 1$ on $\operatorname{supp} g + B(0, 1)$). Since $\|(\varphi\xi_g)(\cdot) - (\varphi\xi_g)(\cdot + s\varepsilon)\|^2_{L^2(dx)}$ goes to 0 when ε goes to 0, uniformly on s on the closed unit ball (again use Plancherel's theorem), (2.13) is proved.

The proof of the theorem will be achieved once we prove Lemma 2.9, and we shall proceed now with this proof.

Let $M \in \mathbb{N}^*$ and let $\psi_M \in \mathscr{C}_c^\infty(\mathbb{R})$ be such that

$$\psi_M(t) = t \qquad \text{for } t \in [-M, M], \quad |\psi_M| \leqslant M+1, \quad |\psi'_M| \leqslant 1$$

and

$$\operatorname{supp}(\psi_M) \subset [-3M, 3M]$$

Define φ_M by $\varphi_M := \psi_M(1/\varphi)$ if $\varphi \neq 0$ and $\varphi_M := 0$ if $\varphi = 0$. Then $\varphi_M \in H^1_{loc}$ and

$$\nabla\varphi_M := -\frac{\nabla\varphi}{\varphi^2}\,\psi'_M\left(\frac{1}{\varphi}\right) \quad \text{if} \quad \varphi \neq 0$$

$$\nabla\varphi_M := 0 \quad \text{if} \quad \varphi = 0$$

since $\psi'_M(1/\varphi) = 0$ on $\{\varphi \leqslant 1/3M\}$. Noticing that $|\nabla\varphi_M| \leqslant |\nabla\varphi|\ 1/\varphi^2\ \mathbb{1}_{|\varphi| > 1/3M}$, we get $\varphi_M \in H^1_{loc}$, $\varphi \in H^1_{loc}$, $\varphi_M \nabla\varphi \in L^2_{loc}(dx)$ since φ_M is bounded, and $\varphi\nabla\varphi_M \in L^2_{loc}(dx)$ since

$$|\varphi\nabla\varphi_M| \leqslant |\nabla\varphi|\,\frac{1}{\varphi}\,\mathbb{1}_{|\varphi| \geqslant 1/3M} \leqslant 3M\,|\nabla\varphi|$$

So according to corollary 2.6ii in the classical case

$$\varphi\,\varphi_M \in H^1_{loc} \quad \text{and} \quad \nabla(\varphi\,\varphi_M) = \varphi_M\nabla\varphi - \frac{\nabla\varphi}{\varphi}\,\psi'_M\left(\frac{1}{\varphi}\right) \tag{2.15}$$

where by convention $\nabla\varphi/\varphi = 0$ when $\varphi = 0$. But again

- φ and $\varphi\,\varphi_M$ are in H^1_{loc}
- $\varphi\,\varphi_M\nabla\varphi \in L^2_{loc}(dx)$ since $|\varphi\,\varphi_M| \leqslant (M+1)/M$ (hence bounded)
- $\varphi\nabla(\varphi\,\varphi_M) \in L^2_{loc}(dx)$ since $\varphi\nabla(\varphi\,\varphi_M) = \varphi\,\varphi_M\nabla\varphi - \nabla\varphi\psi'_M(1/\varphi)$ and both terms are in $L^2_{loc}(dx)$.

It follows that $\varphi^2\,\varphi_M \in H^1_{loc}$ and $\nabla(\varphi^2\,\varphi_M) = 2\,\varphi\,\varphi_M\nabla\varphi - \nabla\varphi\,\psi'_M(1/\varphi)$. In particular we can use the usual integration by parts formula to get

$$\forall g \in C_c^\infty \int \nabla_i g\,\varphi_M\,\varphi^2\,dx = -2\int g\,\varphi_M\,\frac{\nabla_i\varphi}{\varphi}\,\varphi^2\,dx + \int g\,\frac{\nabla_i\varphi}{\varphi^2}\,\psi'_M\left(\frac{1}{\varphi}\right)\varphi^2\,dx$$

In addition $\nabla\varphi_M \in L^2_{loc}(dx)$ since $|\varphi\nabla\varphi_M| \leqslant 3M\,|\nabla\varphi| \in L^2_{loc}(dx)$.

It follows

$$\varphi_M \in H^1_{loc}(\varphi^2\,dx) \qquad \text{and} \qquad \bar{\nabla}\varphi_M = -\nabla\varphi_M = \frac{\nabla\varphi}{\varphi^2}\,\psi'_M\left(\frac{1}{\varphi}\right) \tag{2.16}$$

Though we did not define the local space $H^1_{loc}(\varphi^2\,dx)$, the above sentence is clear. Thanks to (2.16) and corollary 2.6ii again, we have

- $\varphi_M \in H^1_{loc}(\varphi^2\,dx)$, $f \in H^1(\varphi^2\,dx)$ and is compactly supported
- $f\bar{\nabla}\varphi_M \in L^2(\varphi^2\,dx)$ since f is bounded with compact support
- $\varphi_M\bar{\nabla}f \in L^2(\varphi^2\,dx)$ since φ_M is bounded.

So $f\varphi_M \in H^1(\varphi^2\,dx)$ and $\bar{\nabla}(f\varphi_M) = \varphi_M\bar{\nabla}f + f\bar{\nabla}\varphi_M$. We thus may apply (2.3), which yields for $i \in \{1, ..., d\}$ and $g \in \mathscr{C}_c^\infty$

$$\int \bar{\nabla}_i fg\varphi_M\varphi^2\,dx - \int fg\,\frac{\nabla_i\varphi}{\varphi^2}\,\psi'_M\left(\frac{1}{\varphi}\right)\varphi^2\,dx$$
$$= -\int f\varphi_M\,\nabla_i g\varphi^2\,dx - 2\int fg\varphi_M\,\frac{\nabla_i\varphi}{\varphi}\,\varphi^2\,dx \tag{2.17}$$

Now let M go to infinity. We can pass to the limit in (2.17) thanks to the following facts

- φ_M converges dx-a.e. to $1/\varphi\,\mathbb{1}_{\varphi\neq 0}$ and $\psi'_M(1/\varphi)$ converges dx-a.e. to 1
- $|\varphi\varphi_M| \leqslant (M+1)/M \leqslant 2$ so that

$$\bar{\nabla}_i fg\varphi_M\varphi^2| \leqslant 2\,|\varphi\,\bar{\nabla}_i f|\,\|g\|_\infty\,\mathbb{1}_{\operatorname{supp} g} \in L^1(dx)$$

- $|fg\,\nabla_i\varphi\psi'_M(1/\varphi)| \leqslant |f\,\nabla_i\varphi|\,\|g\|_\infty\,\mathbb{1}_{\operatorname{supp} g} \in L^1(dx)$
- $|f\varphi_M\,\nabla_i g\,\varphi^2 \leqslant 2\,|f\varphi|\,\|\nabla_i g\|_\infty\,\mathbb{1}_{\operatorname{supp} g} \in L_1(dx)$
- $|fg\varphi_M\varphi\,\nabla_i\varphi| \leqslant 2\,|f\nabla_i\varphi|\,\|g\|_\infty\,\mathbb{1}_{\operatorname{supp} g} \in L^1(dx)$

The formula obtained by the limiting procedure is exactly Lemma 2.9. ∎

Remark 2.11. The previous proof, though technical and perhaps a little bit tedious, is purely analytic and elementary. It should be underlined that the proofs in [RZ94] or in [Son94b] required more sophisticated material and a touch of Probability theory in both cases.

Theorem 2.7 tells that $(\mathscr{E}_\varphi, H^1(\varphi^2\,dx))$ is the minimal closed extension of $(\mathscr{E}_\varphi, \mathscr{C}_c^\infty)$. According to [AKR90] and [Tak92], it is also the maximal closed extension, hence the unique closed markovian extension, i.e. Markov uniqueness holds, provided $\varphi \in H^1_{loc}$ and $\varphi \neq 0$ dx-a.e., or $\varphi \in H^1$. Another consequence is that $(\mathscr{E}_\varphi, H^1(\varphi^2\,dx))$ is regular (i.e. $\mathscr{C}_c \cap H^1(\varphi^2\,dx)$ is a core for the form), thus associated with a $\varphi^2\,dx$ symmetric Hunt process, which is actually a diffusion process since $(\mathscr{E}_\varphi, H^1(\varphi^2\,dx))$ is local. (see [RZ92] and [ARZ93a] for more details).

A natural question to ask is thus the following: is it possible to prove (at least the major part of) the results of this section, by purely probabilistic methods? The answer to this question is the aim of the next sections.

3. Singular Diffusion Processes via Relative Entropy

Because probabilists prefer to deal with Probability measures, we will assume in this section that

$$\varphi \in H^1 \qquad \text{and} \qquad \int \varphi^2 \, dx = 1 \tag{3.1}$$

The passage to a local condition will be done later on.

We consider the operator S_φ defined on $\mathscr{C}_c^\infty$ by

$$S_\varphi = \frac{1}{2} \Delta + \frac{\nabla \varphi}{\varphi} \cdot \nabla \tag{3.2}$$

and shall study the martingale problem $\mathscr{M}(S_\varphi, C_c^\infty)$ starting from $\varphi^2 \, dx$. From the stochastic calculus viewpoint, the difficulty lies in the fact that $\nabla\varphi/\varphi$ can be a very irregular drift, out of reach of standard result such as Novikov or Kazamaki criteria. But if we remark that

$$\int \left| \frac{\nabla \varphi}{\varphi} \right|^2 \varphi^2 \, dx < +\infty \tag{3.3}$$

(letting $\nabla\varphi/\varphi := 0$ on $\{\varphi = 0\}$) hypothesis 3.1 becomes a "finite energy condition" familiar to specialists of Nelson's stochastic mechanics, and actually almost all problems have already been solved by "entropic methods". Let we first recall the definition of relative entropy (or Kullback information).

Definition 3.1. *Let P and Q be two probability measures on the same space. The relative entropy of Q with respect to P, denoted by $H(Q, P)$, is*

$$\begin{cases} H(Q, P) = \int Z \ln Z \, dP & \textit{if } Q \ll P, \quad Z = dQ/dP, \quad \textit{and} \quad Z \ln Z \in L^1(P) \\ H(Q, P) = +\infty & \textit{otherwise} \end{cases}$$

Our framework in this section will be the following:

- $\Omega = \mathscr{C}^o([0, T], \mathbb{R}^d)$ (for $T > 0$) is the Wiener space, equipped with its usual structure $(\Omega, X_t, \mathscr{F}_t)$.
- if μ is a probability measure on $\mathbb{R}^d$, W_μ denotes the Wiener measure such that $W_\mu \circ X_o^{-1} = \mu$. When $\mu = \varphi^2 \, dx$ we write W_φ instead of $W_{\varphi^2 \, dx}$.
- usual universal completions and right modifications are assumed.

In the sequel the following stopping times will be very important:

$$\tau_n = \inf\left\{t \geqslant 0 \Big/ \int_o^t \left|\frac{\nabla\varphi}{\varphi}\right|^2 (X_s)\, ds \geqslant n\right\}, \qquad \tau = \sup_n \tau_n \tag{3.4}$$

The first result we state is an existence theorem due to Carlen [Car84] (also see [FT84], [Car85], [MZ85], [Zhe85] and many others), and recently extended in [CL94], [CL95a], and [CL95b] to more general contexts. The precise statement below is a consequence of [CL94].

THEOREM 3.2. *The process*

$$Z_t^\varphi = \exp\left(\int_o^{t\wedge\tau} \frac{\nabla\varphi}{\varphi}(X_s)\cdot dX_s - \frac{1}{2}\int_o^{t\wedge\tau} \left|\frac{\nabla\varphi}{\varphi}(X_s)\right|^2 ds\right)$$

is a $\mathscr{F}_t$, W_φ martingale, and the probability measure $Q_\varphi = Z_T^\varphi W_\varphi$ satisfies

(i) Q_φ *is stationary i.e.* $Q_\varphi \circ X_t^{-1} = \varphi^2\, dx\ \forall t \in \neq [0, T]$

(ii) $$H(Q_\varphi, W_\varphi) = \frac{1}{2} E^{Q_\varphi}\left[\int_o^T \left|\frac{\nabla\varphi}{\varphi}(X_s)\right|^2 ds\right] = \frac{T}{2}\int |\nabla\varphi|^2\, dx < +\infty$$

(iii) Q_φ *is a solution to* $\mathscr{M}(S_\varphi, C_c^\infty)$

(iv) Q_φ *is a strongly markovian probability measure.*

Indeed, to apply [CL94], the only point to be checked is that $\varphi^2\, dx$ satisfies the weak stationary Fokker Plack equation $S_\varphi^* v = 0$ which is immediate.

The other point to be precised is the meaning of (iii). We shall say that Q satisfies the martingale problem $\mathscr{M}(S_\varphi, \mathscr{C}_c^\infty)$ if for all $f \in \mathscr{C}_c^\infty$

$$\begin{cases} \displaystyle\int_0^T \left(\frac{\nabla\varphi}{\varphi}\cdot\nabla f\right)(X_s)\, ds \text{ is } Q \text{ a.s. finite} \\[2ex] \displaystyle C_t^f = f(X_t) - f(X_0) - \int_o^t (S_\varphi f)(X_s)\, ds \text{ is a } Q \text{ (continuous) local martingale} \end{cases} \tag{3.5}$$

Since $\mathscr{C}_c^\infty$ is an algebra, it follows (see [Jac79] 13.42 or [DM87]) that $\langle C^f\rangle_t = \int_o^t \Gamma(f)(X_s)\, ds$, where $\Gamma(f) = S_\varphi f^2 - 2f S_\varphi f = |\nabla f|^2$, i.e. the local characteristics of $f(X_t)$, (which is a semi-martingale) are known. Now standard Girsanov theory tells us that Q_φ is a solution to $\mathscr{M}(S_\varphi, \mathscr{C}_c^\infty)$ with $\tau_n \wedge \inf\{t \geqslant 0 \,/\, |X_t| \geqslant n\}$ as a localizing sequence of stopping times.

The next result is an uniqueness theorem, which is a consequence of Th. 12.57(a) in [Jac79], according to the preceding discussion.

THEOREM 3.3. *Let Q be a solution of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ such that $Q \circ X_o^{-1} = \varphi^2\,dx$. If $Q(\tau < +\infty) = 0$ then $Q = Q_\varphi$.*

In particular, Q_φ is the only stationary solution of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ such that $Q \circ X_o^{-1} = \varphi^2\,dx$.

Let us return for a moment to the Dirichlet form $(\mathcal{E}_\varphi, H_o^1(\varphi))$ which is regular and local. According to [Fuk80] chap 6, it is thus associated to a $\varphi^2\,dx$-symmetric diffusion process Q_φ^o, which is in particular a stationary solution of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ as above.

Consequently, $Q_\varphi^o = Q_\varphi$ and Q_φ is thus $\varphi^2\,dx$-symmetric. But as we said before, Theorems 3.2 and 3.3 can be considerably improved, including cases where Fukushima's theory is no more valid. It is thus interesting to get a direct proof of the symmetry of Q_φ. Here again one can use "entropic" arguments, due to H. Föllmer [Föl84].

Indeed, let r denote the time reversal operator

$$\begin{aligned} r\colon \Omega &\to \Omega \\ X &\to (t \to X_{T-t}) \end{aligned} \tag{3.6}$$

Relative entropy is preserved by r, i.e., $H(Q_\varphi \circ r^{-1}, W_\varphi \circ r^{-1}) = H(Q_\varphi, W_\varphi)$ and it follows that $Q_\varphi \circ r^{-1}$ has finite relative entropy with respect to W_{φ_T} (where $\varphi_T = W_\varphi \circ X_T^{-1}$) up to any time $t < T$. Furthermore, the "dual" drift $\hat{B}(t, x)$ (we are in the markovian case) satisfies the duality equation (see [Föl88] 2.13)

$$\hat{B}(t, x) + \frac{\nabla\varphi}{\varphi}(x) = \nabla(\ln \varphi^2)(x) = 2\,\frac{\nabla\varphi}{\varphi}(x) \tag{3.7}$$

Hence $\hat{B} = \nabla\varphi/\varphi$, so that $Q_\varphi \circ r^{-1}$ is again a solution of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ which is $\varphi^2\,dx$-stationary. It follows that $Q_\varphi \circ r^{-1} = Q_\varphi$, i.e. Q_φ is $\varphi^2\,dx$-symmetric.

But one can obtain another very nice property of Q_φ, stated in the next

THEOREM 3.4. *Let $\mathcal{A}_{\varphi,H}$ denote the set of Q probability measure on Ω such that $Q \circ X_t^{-1} = \varphi^2\,dx$ for all $t \in [0, T]$ and $H(Q, W_\varphi) < +\infty$. Then*

(i) *Q_φ is the unique markovian $\varphi^2\,dx$-symmetric element of $\mathcal{A}_{\varphi,H}$.*

(ii) *Furthermore, if $\int (\ln|\varphi|)\,\varphi^2\,dx < +\infty$ then $H(Q_\varphi, W_\varphi) = \inf_{Q \in \mathcal{A}_{\varphi,H}} H(Q, W_\varphi)$.*

Before we proceed with the proof, let we recall the following properties of $\mathcal{A}_{\varphi,H}$ which are obtained in [CL94] (in particular Th. 5.3)

PROPOSITION 3.5. (i) *There exists an unique* $Q_\varphi^* \in \mathscr{A}_{\varphi, H}$ *such that* $H(Q_\varphi^*, W_\varphi) = \inf_{Q \in \mathscr{A}_{\varphi,H}} H(Q, W_\varphi)$

(ii) *Any markovian element of* $\mathscr{A}_{\varphi, H}$ *is a solution to a martingale problem* $\mathscr{M}(S_{\varphi, B}, \mathscr{C}_c^\infty)$ *where* $S_{\varphi, B} = \frac{1}{2}\Delta + B \cdot \nabla$ *and* $B \in L^2(\varphi^2\, dx)$.

(iii) *Conversely, for any* $B \in L^2(\varphi^2\, dx)$ *such that* $S^*_{\varphi, B}(\varphi^2\, dx) = 0$, *the measure* Q_B *defined by* $Q_B = Z_T^B W_\varphi$ *where*

$$\begin{cases} Z_T^B = \exp\left(\int_0^{t \wedge \tau_B} B(X_s) \cdot dX_s - \frac{1}{2}\int_0^{t \wedge \tau_B} |B|^2 (X_s)\, ds\right) \\ \tau_B = \inf\left\{t \geqslant 0 \Big/ \int_0^t |B|^2 (X_s)\, ds = +\infty\right\} \end{cases}$$

is an element of $\mathscr{A}_{\varphi, H}$.

(iv) Q_φ^* *is markovian and its drift* B_φ^* *defined in* (ii) *belongs to the* $L^2(\varphi^2\, dx)$ *closure of the set* $\{\nabla f, f \in \mathscr{C}_c^\infty\}$, *denoted by* $H_o^{-1}(\varphi)$.

(v) *The set* $\{B \in L^2(\varphi^2\, dx) / S^*_{\varphi, B}(\varphi^2\, dx) = 0\}$ *is the affine space* $B_\varphi^* + (H_o^{-1}(\varphi))^\perp$, *where* $\perp$ *stands for the orthogonality in* $L^2(\varphi^2\, dx)$.

We shall use part of this proposition in the proof of Theorem 3.4.

Proof of Theorem 3.4. (i) We shall show that Q_φ is the unique markovian $\varphi^2\, dx$-symmetric element of $\mathscr{A}_{\varphi, H}$. Let Q be such an element. It is immediate that the associated $S_{\varphi, B}$ is $\varphi^2\, dx$-symmetric, by using the martingale problem. Since S_φ is also $\varphi^2\, dx$-symmetric, we get by elementary computations

$$\forall f, g \in \mathscr{C}_c^\infty \int (g \nabla f - f \nabla g) \cdot \left(B - \frac{\nabla p}{p}\right) \varphi^2\, dx = 0$$

But $g \nabla f - f \nabla g = \nabla(f) - 2f \nabla g$ and $\int \nabla(gf) \cdot (B - \nabla\varphi/\varphi)\, \varphi^2\, dx = 0$ according to Proposition 3.5v, so that $\int f \nabla g \cdot (B - \nabla\varphi/\varphi)\, \varphi^2\, dx = 0$. If we choose $g = \psi_M$ as in the proof of Lemma 2.9, and pass to the limit, we get $B = \nabla\varphi/\varphi$. The results follows from the uniqueness Theorem 3.3

(ii) The definition 3.1 can be extended to positive measures (not necessarily bounded), and in particular if $|\int (\ln|\varphi|)\, \varphi^2\, dx| < +\infty$, one has for $Q \in \mathscr{A}_{\varphi, H}$

$$-\infty < H(Q, W_{dx}) = H(Q, W_\varphi) - H(\varphi^2\, dx, dx) < +\infty$$

Furthermore

$$H(Q_\varphi^*, W_{dx}) = \inf_{Q \in \mathscr{A}_{\varphi,H}} H(Q, W_{dx}) \tag{3.8}$$

But here again H is time reversal invariant and so

$$H(Q_\varphi^*, W_{dx}) = H(Q_\varphi^* \circ r^{-1}, W_{dx} \circ r^{-1}) = H(Q_\varphi^* \circ r^{-1}, W_{dx})$$

since W_{dx} is dx-symmetric. So $Q_\varphi^* \circ r^{-1} \in \mathscr{A}_{\varphi, H}$ and realizes the infimum in (3.8), hence $Q_\varphi^* \circ r^{-1} = Q_\varphi^*$. The results follows from the first part of the theorem. ∎

Remark 3.6. (i) According to Proposition 3.5v, we could prove that Q_φ is the minimizing measure in $\mathscr{A}_{\varphi, H}$ for relative entropy if we would be able to show directly that $\nabla\varphi/\varphi$ belongs to $H_o^{-1}(\varphi)$. Notice that this result follows from Theorem 2.7 under the assumption $\ln |\varphi| \in L^2(\varphi^2\, dx)$ (see [BR94b] Th. 2.8 and Section 6 of the present paper).

(ii) Notice that hypothesis $\int (\ln |\varphi|)\, \varphi^2\, dx < +\infty$ is satisfied when $\int |x|^2\, \varphi^2\, dx < +\infty$, i.e. X_o has a second order moment. Indeed if $\int |x|^2\, \varphi^2\, dx < +\infty$, one can use the Gross Log-Sobolev inequality for the Gaussian law dv (see [Gro93])

$$-\infty < \int f^2 \ln |f|\, dv \leqslant \int |\nabla f|^2\, dv + \|f\|^2_{L^2(dv)} \ln \|f\|_{L^2(dv)}$$

with $f = \varphi \exp |x|^2/4$. Elementary computations lead to

$$-\infty < \int \varphi^2 \ln |\varphi|\, dx \leqslant \int |\nabla\varphi|^2\, dx + C^{ste} < +\infty$$

If $\int |x|^2\, \varphi^2\, dx < +\infty$, the derivation of the previous section can also be obtained by replacing the Wiener measure by the Ornstein-Uhlenbeck one whichg in this case is more natural.

4. Singular Diffusion Processes: Markov Kernels and Decompositions

In this section again we assume that $\varphi \in H^1(\mathbb{R}^d)$ and $\int \varphi^2\, dx = 1$, and we put $\nabla\varphi/\varphi(x) = 0$ on $\{\varphi = 0\}$.

We will first build a nice version of φ.

Indeed, it is well known that we can choose a version of φ which is finely continuous outside a polar Brownian set $\mathscr{N}_1$, or equivalently (see [Fuk80]) which is quasi-continuous for the usual (Newtonian) 1-capacity associated to the form $(\mathscr{E}, H^1)$. Such a version is called "précisée" in [MZ85]. But we need to modify again this version. First of all, we modify

φ by choosing $\varphi(x)=0$ on $\mathcal{N}_1$. Now denote by $\mathcal{N}$ the (nodal) set $\{\varphi=0\}$ (in particular, $\mathcal{N}\supset\mathcal{N}_1$). It is known (see e.g. [BG68] Th. II4.8) that

$$t\to\varphi(X_t) \text{ is } W_\mu\text{-a.s. right continuous for all } \mu \text{ such that } \mu(\mathcal{N}_1)=0 \tag{4.1}$$

It immediately follows that if $x\in\mathcal{N}^r$ (the set of regular points of $\mathcal{N}$) and $x\notin\mathcal{N}_1$, then $\varphi(x)=0$. Hence $\mathcal{N}^r\subset\mathcal{N}$, i.e. $\mathcal{N}$ is finely closed, for the Brownian fine topology. But this version is not nice enough.

We next introduce as in (3.4)

$$\tau_k=\inf\left\{t\geqslant 0\Big/\int_o^t\left|\frac{\nabla\varphi}{\varphi}\right|^2(X_s)\,ds\geqslant k\right\},\quad \tau=\tau_\infty=\sup_k\tau_k \tag{4.2}$$

and the following two $(\mathscr{F}_t, W_x)$ supermartingales

$$\begin{cases} Z_t^\varphi=\exp\left(\displaystyle\int_o^{t\wedge\tau}\frac{\nabla\varphi}{\varphi}(X_s)\cdot dX_s-\frac{1}{2}\int_o^{t\wedge\tau}\left|\frac{\nabla\varphi}{\varphi}(X_s)\right|^2 ds\right)\\ \bar{Z}_t^\varphi=\exp\left(\displaystyle\int_o^{t}\frac{\nabla\varphi}{\varphi}(X_s)\cdot dX_s-\frac{1}{2}\int_o^{t}\left|\frac{\nabla\varphi}{\varphi}(X_s)\right|^2 ds\right)\mathbb{1}_{t<\tau}\end{cases} \tag{4.3}$$

It is clear that both Z^φ and $\bar{Z}^\varphi$ are (a.s.) right continuous and that $\bar{Z}^\varphi\leqslant Z^\varphi$. Moreover (see [CL94] and its correction):

Proposition 4.1. (i) *Z^φ is a.s. continuous, Z^φ and $\bar{Z}^\varphi$ only differ on the set*

$$\left\{(t,\omega)/t\geqslant\tau(\omega)\quad\text{and}\quad\omega\in\bigcup_{k\geqslant 1}\{\tau=\tau_k\}\right\}$$

The only possible discontinuities of $\bar{Z}^\varphi$ are for $t=\tau(\omega)$ and $\omega\in\bigcup_{k\geqslant 1}\{\tau=\tau^k\}$.

(ii) *$\bar{Z}^\varphi$ is a strong multiplicative functional, while in general Z^φ is not.*

(iii) *$W_\varphi(\bigcup_{k\geqslant 1}\{\tau=\tau_k\})=0$, hence Z^φ and $\bar{Z}^\varphi$ coïncide W_φ-a.s., and are actually $(\mathscr{F}_t, W_\varphi)$ martingales (recall Th. 3.2).*

In the sequel, we denote by $\mathcal{N}'$ the subset of $\mathbb{R}^d$ such that

$$\mathbb{R}^d-\mathcal{N}'=\left\{x\in\mathbb{R}^d\;\middle|\;\begin{array}{l}Z^\varphi\text{ is a }(\mathscr{F}_t, W_x)\text{ martingale (i.e. }E^{W_x}(Z_T^\varphi)=1\\ \text{and }W_x(\bigcup_{k\geqslant 1}\{\tau=\tau_k\})=0\text{ (i.e. }Z^\varphi=\bar{Z}^\varphi\ W_x\text{-p.s.)}\end{array}\right\} \tag{4.4}$$

(recall that we are working on $\Omega=\mathscr{C}^o([0,T],\mathbb{R}^d)$ i.e. we define $X_t=X_T$ for $t\geqslant T$) Since $x\to W_x$ is measurable, $\mathcal{N}'$ is a Borel set. So according to

Th. I.10.19 of [BG68] one can find an increasing sequence of compact subsets K_n of $\mathcal{N}'$ such that $T_{K_n} \searrow T_{\mathcal{N}'}$ where T_A is the hitting time of A. Applying the strong Markov property of W_x with stopping time T_{K_n}, and Lebesgue's bounded convergence theorem, it is not difficult to check that $\mathcal{N}'$ is finely closed (for the Brownian fine topology). Furthermore, $\mathcal{N}'$ is $\varphi^2\,dx$-negligible thanks to Proposition 4.1(iii). Finally, define

$$\begin{cases} \sigma_n = \inf\{t \geqslant 0 \,/\, \varphi(X_t) \notin [1/n, n]\} \\ \sigma = \sup_n \sigma_n \\ \sigma' = \sigma \wedge \inf\{t \geqslant 0 \,/\, X_t \in \mathcal{N}'\} \end{cases} \tag{4.5}$$

$$\text{and} \quad \begin{cases} Q_x = Z_T^{\varphi}\, W_x & \text{if} \quad x \notin \mathcal{N} \cup \mathcal{N}' \\ Q_x = \delta_{\{x\}} & x \in \mathcal{N} \cup \mathcal{N}' \end{cases} \tag{4.6}$$

where $\delta_{\{x\}}$ is the Dirac mass on the constant path $X_t = x\ \forall t \in [0, T]$. Our first result is

Theorem 4.2. (i) $\forall x \notin \mathcal{N} \cup \mathcal{N}'\ Q_x(\sigma' < +\infty) = 0$

(ii) $(Q_x)_{x \in \mathbb{R}^d}$ *is a strong Markov family of probability measures, such that* $Q_\varphi = \int Q_x\, \varphi^2\, dx$.

In particular we can modify φ by putting $\varphi(x) = 0$ for $x \in \mathcal{N}'$ without changing Q_x. The above theorem provides a nice realization of the disintegration of Q_φ, i.e. if we choose the good φ, the nodal set $\mathcal{N}$ cannot be reached (except if one starts from a nodal point), and outside of $\mathcal{N}$ Q_x has the good Girsanov density.

Proof of Theorem 4.2. Since $\mathcal{N} \cup \mathcal{N}'$ is $\varphi^2\,dx$-negligible, the equality $Q_\varphi = \int Q_x\, \varphi^2\, dx$ is immediate. Indeed, Q_x satisfies the martingale problem $\mathcal{M}(\mathcal{C}_c^\infty, S_\varphi)$ for $x \notin \mathcal{N} \cup \mathcal{N}'$ and so does $\int Q_x\, \varphi^2\, dx$. Furthermore, $Q_x(\tau < +\infty) = 0$ for $x \notin \mathcal{N} \cup \mathcal{N}'$, and we may apply Theorem 3.3. In order to prove the rest of the theorem, we first recall the

Proposition 4.3. $Q_\varphi(\sigma < +\infty) = 0$ (*see* [MZ85]).

Proposition 4.3 implies that for $\varphi^2\,dx$-almost all x, $Q_x(\sigma < +\infty) = 0$. We want more, since we want that $Q_x(\sigma' < +\infty) = 0$ for the x's which do not belong to $\mathcal{N} \cup \mathcal{N}'$. To this end, first remark that

$$\text{if} \quad x \notin \mathcal{N} \cup \mathcal{N}', \quad \inf\{t \geqslant 0 \,/\, X_t \in \mathcal{N}'\} = +\infty\ Q_x\text{-a.s.}$$

which is an immediate consequence of the strong Markov property of W_x, since $\mathcal{N}'$ is finely closed. Since $Q_x << W_x$ for these x's, we only have to prove that

$$\text{if } x \notin \mathcal{N} \cup \mathcal{N}',\ Q_x(\sigma < +\infty) = 0$$

Assume that $Q_x(\sigma < +\infty) = \alpha > 0$, which is equivalent to $Q_x(\sigma \wedge \tau < +\infty) = \alpha > 0$ since $\tau = +\infty$ Q_x-a.s. according to (4.4). We can then find $t_o > 0$ and $n \in \mathbb{N}^*$ such that for all $t \leqslant t_o$

$$Q_x(t < \sigma_n \wedge \tau_n < \sigma \wedge \tau < +\infty) \geqslant \frac{\alpha}{2} > 0 \tag{4.7}$$

Indeed, thanks to (4.4), $\tau > \tau_n$ and $\tau > 0$ Q_x-a.s., and thanks to (3.4.) $\sigma > \sigma_n > 0$ (for n large enough). Then we apply the multiplicativity of $\bar{Z}^{\varphi}$.

$$\begin{aligned}
&Q_x(t < \sigma_n \wedge \tau_n < \sigma \wedge \tau < +\infty)\\
&\quad = \int \mathbb{1}_{1 < \sigma_n \wedge \tau_n \wedge T} \bar{Z}^{\varphi}_t \bar{Z}^{\varphi}_{(\sigma \wedge \tau) - t}(\theta_t(\omega))\, \mathbb{1}_{\sigma \wedge \tau < +\infty}(\theta_t(\omega))\, dW_x\\
&\quad = \int \mathbb{1}_{\sigma_n \wedge \tau_n \wedge T} \bar{Z}^{\varphi}_t\\
&\qquad \times \left(\int \bar{Z}^{\varphi}_{(\sigma \wedge \tau) - t}(\omega')\, \mathbb{1}_{\sigma \wedge \tau < +\infty}(\omega')\, dW_{X_t(\omega)}(\omega') \right) dW_x(\omega)\\
&\quad \leqslant \int \mathbb{1}_{t < \sigma_n \wedge \tau_n \wedge T} h(X_t)\, \bar{Z}^{\varphi}_{\sigma_n \wedge \tau_n}\, dW_x \qquad (4.8)\\
&\qquad \text{where } h(x) = \int \bar{Z}^{\varphi}_{(\sigma \wedge \tau) - t}(\omega')\, \mathbb{1}_{\sigma \wedge \tau < +\infty}(\omega')\, dW_x(\omega')\\
&\quad \leqslant \int_{\mathbb{R}^4} h(y)\, p^n_t(x, y)\, dy
\end{aligned}$$

where $p^n_t(x, \cdot)$ is the density of $(\bar{Z}^{\varphi}_{\sigma \wedge \tau \wedge T} W_x) \circ X_t^{-1}$ with respect to Lebesgue's measure (which exists since $(\bar{Z}^{\varphi}_{\sigma \wedge \tau \wedge T} W_x)$ is equivalent to W_x).

If $\varphi \neq 0$ dx-a.s., the proof is finished, since $\varphi^2\, dy$ is equivalent to dy and $h(y) = Q_y(\sigma \wedge \tau < +\infty)$ for $\varphi^2\, dy$-almost all y, i.e. vanishes $\varphi^2\, dy$-a.s. thanks to Proposition 4.3. This leads to a contradiction with (4.7).

It $\{\varphi = 0\}$ is not of Lebesgue's measure equal to 0, (4.7) and (4.8) imply that

$$\forall t \leqslant t_0 \int_N h(y)\, p^n_t(x, y)\, dy \geqslant \frac{\alpha}{2} > 0$$

and since $h \leqslant 1$

$$\forall t \leqslant t_0 \int \bar{Z}^{\varphi}_{\sigma \wedge \tau \wedge T} \mathbb{1}_{X_t \in \mathcal{N}} \, dW_x \geqslant \frac{\alpha}{2} > 0 \tag{4.9}$$

Now recall that $\mathcal{N}$ is finely closed, and $x \in \mathcal{N}^c$ which is finely open i.e.

$$W_x(\inf\{t > 0/X_t \in \mathcal{N}\} = 0) = 0$$

Since W_x and $\bar{Z}^{\varphi}_{\sigma \wedge \tau \wedge T} W_x$ are equivalent, the above also holds for $\bar{Z}^{\varphi}_{\sigma \wedge \tau \wedge T} W_x$, and we obtain a contradiction with (4.9).

We thus have proved Theorem 4.2.i. Part ii now is immediate since $\bar{Z}^{\varphi} = Z^{\varphi}$ is strong multiplicative for $x \notin \mathcal{N} \cup \mathcal{N}'$, and $\mathcal{N} \cup \mathcal{N}'$ cannot be attained if we start from such an x. ∎

In addition

Proposition 4.4. *Let*

$$\mathcal{N}'' = \mathcal{N} \cup \mathcal{N}' \cup \left\{ x \in \mathbb{R}^d / \exists t > 0 \middle/ E^{Q_x} \left(\int_o^t \left| \frac{\nabla \varphi}{\varphi} \right|^2 (X_s) \, ds \right) = +\infty \right\}$$

Then $\mathcal{N}''$ is $\varphi^2 \, dx$-negligible, finely closed (for both the Brownian or the $(Q_x)_x$ fine topology), and if $x \notin \mathcal{N}''$ then $Q_x(\sigma' \wedge \inf\{s \geqslant 0/X_s \in \mathcal{N}''\} < +\infty) = 0$. Hence we can modify Q_x, by setting $Q_x = \delta_{\{x\}}$ if $x \in \mathcal{N}''$ in (4.6).

The proof is immediate and left to the reader.

During the proof of Proposition 4.3, appeared $\ln \varphi$. In most of the works concerned with the problem we are dealing with (or similar existence and uniqueness problems), the results are derived by studying precisely the behavior of the process $\ln \varphi(X_t) - \ln \varphi(X_o)$ (see [Tak92], [Son94b] for the symmetric case, also [ARZ93a], and [MZ84], [Car85], [Zhe85] in the framework of Nelson's stochastic mechanics). In the symmetric case, one can use Fukushima's decomposition provided $\ln \varphi$ belongs to the domain of the form induced by $(Q_x)_x$ (in particular the integrability condition $\int (\ln \varphi)^2 \varphi^2 \, dx < +\infty$ is required, condition which is forgotten in some papers).

This decomposition can be recovered, in a probabilistic framework, by using a recent extension of Ito formula due to Föllmer and Protter [FP94] (see [Fra95a]).

Extension to $\mathscr{C}^o(\mathbb{R}^+, \mathbb{R}^d)$. Of course all we have done can easily be extended to the whole paths space $\mathscr{C}^o(\mathbb{R}^+, \mathbb{R}^d)$. It suffices to apply the above results, with the increasing sequence of times $T_n = n$, by defining $\mathcal{N} \cup \mathcal{N}' = \bigcup_n (\mathcal{N} \cup \mathcal{N}')_{T_n}$ and Q_x as the Föllmer measure associated with the martingale (but supermartingale up to and including $T=+\infty$) Z^{φ} for $x \notin \mathcal{N} \cup \mathcal{N}'$. It can happen that Q_x and W_x are singular, but $Q_x << W_x$ in restriction to $\mathcal{F}_T$ for all $T<+\infty$. (see [Jac79] or [Föl72] for a discussion about the Föllmer measure.)

5. Singular Diffusion Processes and the Associated Dirichlet Forms

In this section we shall discuss the relationship between the Dirichlet forms we introduced in Section 2 and the singular diffusion processes we studied in the previous two sections. In particular we intend to give another proof of Theorem 2.7, by using "quasi only" probabilistic tools. So before completing this new proof, we will not assume that $H_o^1(\varphi^2\,dx)=H^1(\varphi^2\,dx)$.

Let we assume for the moment that (3.1) holds.

We saw in Theorem 3.3 that there exists one and only one stationary (hence symmetric) solution to $\mathcal{M}(\mathcal{C}_c^\infty, S_\varphi)$ (though we worked in $\mathcal{C}([0,T], \mathbb{R}^d)$, Theorem 3.3 extends to any solution of $\mathcal{M}(\mathcal{C}_c^\infty, S_\varphi)$ in the Skorohod space of cadlag paths). In particular if $(\mathcal{E}_\varphi, H)$ is a closed markovian extension of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$, which is associated with a right Markov process, one expects that this process is given by Q_φ. Actually

Proposition 5.1. *There exists an unique quasi-regular markovian closed extension $(\mathcal{E}_\varphi, H_o^1(\varphi^2\,dx))$ of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$ such that the generator A_0 of $(\mathcal{E}_0, H_0^1(\varphi^2\,dx))$ extends $(S_\varphi, \mathcal{C}_c^\infty)$.*

Proof. According to [AM91] Th. 1.7 (add a remark of [Fit89]), a symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is associated with a right symmetric process if and only if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is quasi-regular. Let Q be the law of this process with generator $(A, \mathcal{D}(A))$. Q solves the martingale problem $(\mathcal{M}(\mathcal{D}(A), A)$ and since A is an extension of S_φ, Q also solves the martingale problem $\mathcal{M}(\mathcal{C}_c^\infty, S_\varphi)$. ∎

Unfortunately, the notion of quasi-regularity we shall not discuss here is a generalization of the classical regularity of [Fuk80] (i.e. $\mathcal{D}(\mathcal{E}) \cap \mathcal{C}_c$ is a core) and it seems that to prove quasi-regularity for $(\mathcal{E}_\varphi, H^1(\varphi^2\,dx))$ is not easier than a direct proof of $H_o^1(\varphi^2\,dx)=H^1(\varphi^2\,dx)$, i.e. the above corollary is not really tractable for the proof of Markov uniqueness.

The proof we shall propose now is inspired by [Son94b] with some simplifications.

Theorem 5.2. *If* (3.1) *holds then* $H_o^1(\varphi^2\,dx)=H^1(\varphi^2\,dx)$.

Proof of Theorem 5.2. Let χ_n denote the 1-equilibrium potential of $\{\varphi \notin [1/n, n]\}$, for the form $(\mathscr{E}_\varphi, H^1_o(\varphi^2\,dx))$, i.e. (see [Fuk80] Th. 3.3.1) $\chi_n \in H^1_o(\varphi^2\,dx)$ and $cap_\varphi(\{\varphi \notin [1/n, n]\}) = \mathscr{E}^1_\varphi(\chi_n, \chi_n)$. Since $(\mathscr{E}_\varphi, H^1_o(\varphi^2\,dx))$ is regular, it is known that one can choose $\chi_n(x) = E^{Q_x}(e^{-\sigma_n})$ and that $cap_\varphi(\{\varphi \notin [1/n, n]\}) = E^{Q_\varphi}(e^{-\sigma_n})$ (see [Fuk80] Th. 4.3.5).

Remark 5.3. Though $\operatorname{supp} \varphi^2\,dx \neq \mathbb{R}^d$ a priori, the above results are still true (see e.g. [AM91]).

According to Proposition 4.3 and Lebesgue's dominated convergence theorem

$$\lim_{n \to +\infty} E^{Q_\varphi}(e^{-\sigma_n}) = 0 \tag{5.1}$$

Pick an $f \in H^1(\varphi^2\,dx)$ which is bounded. We shall prove that f can be approximated by a sequence of elements of $H^1_o(\varphi^2\,dx)$.

Since f and χ_n are bounded elements of $H^1(\varphi^2\,dx)$, $f\chi_n \in H^1(\varphi^2\,dx)$ and is bounded (see Fuk80] Th. 1.4.2 ii). Furthermore

$$\bar{\nabla}(f\chi_n) = f\bar{\nabla} X_n + \chi_n \bar{\nabla} f \tag{5.2}$$

which is easily obtained by taking limits in 2.3 since $\chi_n \in H^1_o(\varphi^2\,dx)$ (i.e. one can find a sequence of $\mathscr{C}^\infty_c$ which converges to χ_n in $\mathscr{E}^1_\varphi$ norm).

Lemma 5.4. $\lim_{n \to +\infty} \mathscr{E}^1_\varphi(\chi_n f, \chi_n f) = 0$ *(up to a subsequence)*.

Proof. Since $\mathscr{E}^1_\varphi(\chi_n, \chi_n)$ goes to 0, one can find a subsequence such that χ_n and $\bar{\nabla}\chi_n$ goes to 0, both in $L^2(\varphi^2\,dx)$ and $\varphi^2\,dx$ a.s. Thus since f is bounded,

$$\lim_{n \to +\infty} \int (\chi_n f)^2\, \varphi^2\,dx = 0$$

and

$$\int |\bar{\nabla}(\chi_n f)^2\, \varphi^2\,dx \leqslant 2 \int f^2\, |\bar{\nabla}\chi_n|^2\, \varphi^2\,dx + \int \chi_n^2\, |\bar{\nabla} f|^2\, \varphi^2\,dx$$

which again goes to 0 thanks to Lebesgue's theorem (recall that $\chi_n^2 \leqslant 1$). ∎

In view of Lemma 5.4, it is enough to approximate $(1 - \chi_n) f$ by elements of $H^1_o(\varphi^2\,dx)$. The proof will rely on the following result.

LEMMA 5.5. *Define*

$$H_n^1(\varphi)=\Big\{g\in H^1(\varphi^2\,dx),\ g \text{ bounded} \,/\, g(x)=0 \quad \text{and}$$

$$\bar{\nabla} g(x)=0 \text{ for } dx\text{-almost all } x\notin\Big\{\varphi\in\Big[\frac{1}{n},n\Big]\Big\}\Big\}$$

$$H_n^1=\Big\{g\in H^1,\ g \text{ bounded} \,/\, g(x)=0 \text{ for } dx\text{-almost all } x\notin\Big\{\varphi\in\Big[\frac{1}{n},n\Big]\Big\}\Big\}$$

Then $H_n^1(\varphi)\subset H_n^1$, *and if* $g\in H_n^1(\varphi)$, $\nabla g=\bar{\nabla} g \mathbb{1}_{\varphi\in[1/n,\,n]}=\bar{\nabla} g$.

The proof of Lemma 5.5 is postponed to the end of the proof of Theorem 5.2.

Since $\chi_n\in H_o^1(\varphi^2\,dx)$, which is the domain of a regular form, one knows that $\bar{\nabla}\chi_n=\nabla\chi_n \mathbb{1}_{\varphi\in[1/n,\,n]}$. According to (5.2) the function $(1-\chi_n)f$ belongs to $H_n^1(\varphi)$, hence by Lemma 5.5, one can find a sequence $(\psi_k^n)_{k\geqslant 1}$ of test functions such that $|\psi_k^n|\leqslant\|f\|_\infty$ and

$$\psi_k^n \xrightarrow[k\to+\infty]{} (1-\chi_n)f \text{ in } H^1$$

(take care that ψ_k^n does not need to vanish on $\{\varphi\notin[1/n,n]\}$). But

$$(1-\chi_n)\,\psi_k^n\in H_o^1(\varphi^2\,dx)\cap H_n^1(\varphi)$$

Let we compute the distance between $(1-\chi_n)f$ and $(1-\chi_n)\,\psi_k^n$. First of all

$$\int[(1-\chi_n)\,\psi_k^n-(1-\chi_n)f]^2\,\varphi^2\,dx$$

$$=\int \mathbb{1}_{\varphi\in[1/n,\,n]}[(1-\chi_n)\,\psi_k^n-(1-\chi_n)f]^2\,\varphi^2\,dx$$

$$\leqslant 2\int \mathbb{1}_{\varphi\in[1/n,\,n]}[\psi_k^n-(1-\chi_n)f]^2\,\varphi^2\,dx+2\int|\chi_n\psi_k^n|^2\,\varphi^2\,dx$$

$$\leqslant 2n^2\,\|\psi_k^n-(1-\chi_n)f\|^2_{L^2(dx)}+2\,\|f\|^2_\infty\,\|\chi_n\|^2_{L^2(\varphi^2\,dx)}$$

Next

$$\int |\bar{\nabla}((1-\chi_n)\,\psi_k^n) - \bar{\nabla}((1-\chi_n)\,f)|^2\,\varphi^2\,dx$$

$$= \int |(1-\chi_n)\,\bar{\nabla}\psi_k^n - \bar{\nabla}((1-\chi_n)\,f) - \psi_k^n \bar{\nabla}\chi_n|^2\,\mathbb{1}_{\varphi \in [1/n,\,n]}\,\varphi^2\,dx$$

$$= \int |(1-\chi_n)[\bar{\nabla}\psi_k^n - \bar{\nabla}((1-\chi_n)\,f)] - \chi_n \bar{\nabla}((1-\chi_n)\,f) - \psi_k^n \bar{\nabla}\chi_n|^2$$

$$\times\,\mathbb{1}_{\varphi \in [1/n,\,n]}\varphi^2\,dx$$

$$= \int |(1-\chi_n)[\bar{\nabla}\psi_k^n - \bar{\nabla}((1-\chi_n)\,f)] - \chi_n(1-\chi_n)\,\bar{\nabla}f + (\chi_n f - \psi_k^n)\,\bar{\nabla}\chi_n|^2$$

$$\times\,\mathbb{1}_{\varphi \in [1/n,\,n]}\,\varphi^2\,dx$$

$$\leqslant 3n^2\,\|\bar{\nabla}\psi_k^n - \bar{\nabla}((1-\chi_n)\,f)\|^2_{L^2(dx)}$$

$$+ 3\int |\chi_n \bar{\nabla} f|^2\,\varphi^2\,dx + 3\times 2\,\|f\|^2_\infty\,\|\bar{\nabla}\chi_n\|^2_{L^2(\varphi^2\,dx)}$$

By choosing first n big enough, one can control all terms where k does not appear (thanks to Lemma 5.4, (5.1) and Lebesgue's theorem). One can then choose k large enough for $\|\psi_k^n - (1-\chi_n)\,f\|^2_{H^1}$ to be as small as we want. We just have proved that

$$\mathscr{E}^1_\varphi((1-\chi_n)\,\psi_k^n - (1-\chi_n)\,f,\,(1-\chi_n)\,\psi_k^n - (1-\chi_n)\,f)$$

is arbitrary small, and since $(1-\chi_n)\,\psi_k^n \in H^1_o(\varphi^2\,dx)$, the proof of Theorem 5.2 is finished provided we show Lemma 5.5:

Proof. Let $u \in \mathscr{C}_c^\infty$ and let u_n denote the function $u_n = u/((\varphi \vee 1/n) \wedge n)^2$. Then $u_n \in H^1$ and

$$\nabla u_n = \frac{\nabla u}{((\varphi \vee 1/n) \wedge n)^2} - 2u\,\frac{\nabla \varphi}{((\varphi \vee 1/n) \wedge n)^3}\,\mathbb{1}_{\varphi \in [1/n,\,n]}$$

Let $(h_n^k)_{k \geqslant 1}$ be a sequence of test functions which converges to u_n in H^1. Apply (2.3) in order to get

$$\int h_n^k\,\bar{\nabla} g\,\varphi^2\,dx = -\int g\,\bar{\nabla} h_n^k\,\varphi^2\,dx - 2\int g h_n^k\,\frac{\nabla\varphi}{\varphi}\,\varphi^2\,dx \tag{5.3}$$

But according to what precedes, all functions under integral signs vanish outside of $\{\varphi \in [1/n, n]\}$, and on $\{\varphi \in [1/n, n]\}$, $\varphi^2\, dx$ and dx are equivalent. But

$$h_n^k \mathbb{1}_{\varphi \in [1/n,\, n]} \varphi^2 \xrightarrow[k \to +\infty]{} u \mathbb{1}_{\varphi \in [1/n,\, n]}\, dx \text{ a.s.}$$

$$\nabla h_n^k \mathbb{1}_{\varphi \in [1/n,\, n]} \varphi^2 \xrightarrow[k \to +\infty]{} \left(\nabla u - 2u \frac{\nabla \varphi}{\varphi}\right) \mathbb{1}_{\varphi \in [1/n,\, n]}\, dx \text{ a.s.}$$

and then by taking limits in (5.3), we get

$$\int u\, \overline{\nabla} g\, \mathbb{1}_{\varphi \in [1/n,\, n]}\, dx = \int \nabla u\, g\, \mathbb{1}_{\varphi \in [1/n,\, n]}\, dx = \int \nabla u\, g\, dx$$

The proof of Lemma 5.5, thus of Theorem 5.2, is finished. ∎

Remark 5.6. Though the proof seems to be a little bit more complicated than we expected, we want to underline that all analytical material we used is really elementary and does not refer to the special form of the generator (i.e. to Proposition 2.5 and its consequences). See e.g. [Fra95b] for some extensions.

Theorem 5.2 can now be easily extended to the H^1_{loc} case, i.e.

THEOREM 5.7. *If $\varphi \in H^1_{loc}$, then $H^1_o(\varphi^2\, dx) = H^1(\varphi^2\, dx)$.*

Proof. According to Corollary 2.6iii, it suffices to approximate a $f \in H^1(\varphi^2\, dx)$ which is bounded and compactly supported. Take χ and χ', two $\mathscr{C}_c^\infty$ functions satisfying $\mathbb{1}_{\operatorname{supp} f} \leqslant \chi \leqslant \chi' \leqslant 1$ and $\mathbb{1}_{\operatorname{supp} \chi} \leqslant \chi'$. Then $f = f\chi$ belongs to $H^1((\chi'\varphi)^2\, dx)$. But $\chi'\varphi$ being compactly supported satisfies (3.1) (up to a normalizing constant) i.e. $H^1_o((\chi'\varphi)^2\, dx) = H^1((\chi'\varphi)^2\, dx)$ thanks to Theorem 5.2. Let f_n be a sequence of $\mathscr{C}_c^\infty$ such that f_n converges to f in $H^1((\chi'\varphi)^2\, dx)$ and put $g_n = \chi f_n$. Then

$$\int |f - g_n|^2\, \varphi^2\, dx = \int |\chi f - \chi f_n|^2 (\chi'\varphi)^2\, dx \leqslant \int |f - f_n|^2 (\chi'\varphi)^2\, dx$$

and

$$\int |\overline{\nabla} f - \nabla g_n|^2\, \varphi^2\, dx$$

$$\leqslant 2 \int |\chi \overline{\nabla} f - \chi \nabla f_n|^2 (\chi'\varphi)^2\, dx + 2 \int |f_n|^2\, |\nabla \chi|^2\, (\chi'\varphi)^2\, dx$$

$$\leqslant 2 \int |\bar{\nabla} f - \nabla f_n|^2 \, (\chi' \varphi)^2 \, dx + 2 \int |f_n - f|^2 \, |\nabla \chi|^2 \, (\chi' \varphi)^2 \, dx$$

$$\text{since } f \nabla \chi \equiv 0$$

$$\leqslant C^{ste} \, \|f_n - f\|^2_{H^1((\chi' \varphi)^2 \, dx)}$$

Both quantities are then going to 0 with n, and the proof is finished. ▮

Remark 5.8. We cannot change the reference measure dx into dv (with v the Gaussian standard law) except if we assume $\int |x|^2 \varphi^2 \, dx < +\infty$ (otherwise $\varphi/\sqrt{\gamma}$, where $\gamma \, dx = dv$, does not belong to $H^1(v)$). It seems that this point is overcomed in the proof of [RZ92] Th. 4.1.

COROLLARY 5.9. *If either $\varphi \in H^1$, or $\varphi \in H^1_{loc}$ and $\varphi \neq 0$ dx a.e., Markov uniqueness holds.*

6. INVARIANT MEASURES

In this section we shall use our previous results to study a large class of invariant measures of the Markov process induced on $\Omega = \mathscr{C}^o(\mathbb{R}^+, \mathbb{R}^d)$ by the family $(Q_x)_{x \in \mathbb{R}^d}$ given by (4.4). A similar study is done in [BR94b], by using purely analytical tools. Our results are more complete and are extended to the general diffusion case in [Fra95b].

We assume again that $\varphi \in H^1(\mathbb{R}^d)$ and $\int \varphi^2 \, dx = 1$, and choose the version of φ (resp. Q_x) defined in Th. 4.2 (resp. Prop. 4.4).

Among the invariant measures, the set of the reversible ones is of particular interest. We already know that $\varphi^2 \, dx$ is reversible. We shall see that any reversible measure "looks like" $\varphi^2 \, dx$, and moreover that any stationary measure satisfying an integrability condition is reversible. To this end we shall use the entropy minimization result mentioned in Theorem 3.4ii, but we first drop the condition $\int (\ln |\varphi|) \, \varphi^2 \, dx < +\infty$.

PROPOSITION 6.1. *$H(Q_\varphi, W_\varphi) = \inf_{Q \in \mathscr{A}_{\varphi, H}} H(Q, W_\varphi)$ where $\mathscr{A}_{\varphi,H} = \{Q/Q \circ X_t^{-1} = \varphi^2 \, dx \ \forall t\}$.*

Proof. Proposition 6.1 will be proved if we show that $\nabla\varphi/\varphi$ belongs to $H_o^{-1}(\varphi^2 \, dx)$ (the closure of $\{\nabla f/f \in \mathscr{C}_c^\infty\}$ in $L^2(\varphi^2 \, dx)$). But

$$g_\varepsilon = \ln\left((\varphi \vee \varepsilon) \wedge \frac{1}{\varepsilon}\right) \qquad \text{and} \qquad \nabla g_\varepsilon = \frac{\nabla\varphi}{\varphi} \mathbb{1}_{\varepsilon \leqslant \varphi \leqslant 1/\varepsilon}$$

By taking limits, and since $H_o^1(\varphi^2 \, dx) = H^1(\varphi^2 \, dx)$ (according to Theorem 5.7) the desired result is obtained. ▮

Now let μ be a bounded invariant measure. μ splits into $\mu = \mu \mathbb{1}_{\varphi=0} + \mu \mathbb{1}_{\varphi>0} = \mu_o + \mu_+$. Of course, μ_o is a reversible measure and μ_+ is an invariant measure. In the sequel, we shall study μ_+.

PROPOSITION 6.2. $\mu_+ \ll dx$.

Proof. $Q_{\mu_+} \ll W_{\mu_+}$, thus $Q_{\mu_+} \circ X_t^{-1} \ll W_{\mu_+} \circ X_t^{-1} \ll dx$ for all $t > 0$. ∎

PROPOSITION 6.3. *Denote by ψ^2 the density of μ_+, i.e. $d\mu_+ = \psi^2\, dx$. If $\int |\nabla\varphi/\varphi|^2\, \psi^2\, dx < +\infty$, then $\psi \in H^1$.*

Proof.

$$H(Q_{\mu_+}, W_{\mu_+}) = \frac{1}{2} E^{Q_{\mu_+}} \left[\int_o^T \left| \frac{\nabla\varphi}{\varphi} \right|^2 (X_s)\, ds \right] = \frac{T}{2} \int \left| \frac{\nabla\varphi}{\varphi} \right|^2 \psi^2\, dx$$

(if we restrict the paths to the time interval $[0, T]$.) Thus we can use Föllmer's results on time reversal (see [Föl88] and [Föl84]), which shows that $\nabla\psi$ is dx a.e. well defined and $\nabla\psi/\psi \in L^2(\psi^2\, dx)$ i.e. (since μ_+ is bounded) $\psi \in H^1$. ∎

(See [BR94b] for an analytical proof.)
Finally, we can state

THEOREM 6.4. *Any bounded invariant measure μ satisfying $\int |\nabla\varphi/\varphi|^2\, d\mu < +\infty$ is a reversible measure.*

Proof. Since $\nabla\varphi/\varphi = 0$ on $\{\varphi = 0\}$, the above statement reduced to $\int |\nabla\varphi/\varphi|^2\, d\mu_+ < +\infty$.

It follows from Proposition 6.3 that $d\mu_+ = \psi^2\, dx$ with $\psi \in H^1$. Thanks to Theorem 5.7, $H_o^1(\psi^2\, dx) = H^1(\psi^2\, dx)$, and by using the same argument than in the proof of Proposition 6.1 we obtain that $\nabla\varphi/\varphi \in H_o^{-1}(\psi^2\, dx)$. But this proves, thanks to Proposition 3.5, that $H(Q_{\mu_+}, W_\psi) = \inf_{Q \in \mathscr{A}_{\psi,H}} H(Q, W_\psi)$. But this infimum is attained at a single point, hence $Q_+ = Q_\psi$ according to Proposition 6.1, and we know that Q_ψ is time reversible. ∎

The above proof gives another information: $Q_{\mu_+} = Q_\psi$. So Q_{μ_+} solves the martingale problem $\mathscr{M}(\mathscr{C}_c^\infty, S_\varphi)$ and is time reversible. It follows immediately (see the proof of Theorem 3.4i) that

$$\frac{\nabla\psi}{\psi} = \frac{\nabla\varphi}{\varphi}\, \psi^2\, dx \text{ a.s.}$$

Hence $\varphi^2 - \psi^2$ is locally constant on the interior of the set $\{\psi > 0\}$, i.e. constant on each connected component of the interior of $\{\psi > 0\}$.

This solves the problem of uniqueness of invariant measures (satisfying the finite energy condition, and bounded). This problem is studied in [BR94b] Section 6, with apparently a less complete solution than ours, since the authors assume that $|\nabla\varphi|/\varphi \in L^1_{loc}(dx)$.

Remark 6.5. All what precedes extends to the case of a non necessarily bounded μ. Proposition 6.2 extends to this case without any trouble. If we consider the generalization of Kullback information to non necessarily finite measures, it is well known that

$$H(Q_\mu, W_\mu) = \int H(Q_x, W_x)\, d\mu$$

If $\nabla\varphi/\varphi$ is of finite $\psi^2\, dx$-energy, it is not hard to extend Föllmer time reversal argument (the nature of the time reversed Brownian motion does not depend on the initial measure), hence $\psi \in L^2_{loc}(dx)$ and $\nabla\psi \in L^2(dx)$. The minimality argument which leads to Proposition 6.1 is only connected with Riesz projection theorem on a Hilbert space (see [CL94]), so extends to a non necessarily bounded measure (provided $\mathscr{C}_c^\infty \subset L^2(\mu)$).

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